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QUADRUPLE INTEGRAL EQUATIONS FOR THE CHARGED DISC AND COPLANAR ANNULUS

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Quadruple integral equations for the charged disc and coplanar annulus

by

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ABSTRACT

The axisymmetric potential problem for a plane circular electrode of radius a and a concentric annulus $b < r < c$ $(a < b)$ is formulated in terms of quadruple integral equations for the Hankel transform of the potential, and reduced to a single Fredholm equation. For small or large values of some combinations of the geometrical parameters a, b and c this equation is solved asymptotically.

KEY WORDS & PHRASES: *Quadruple integral equations, Hankel transforms, charged disc, Fredholm integral equation, mixed boundary value problem*

1. INTRODUCTION

The potential problem is formulated as follows: Find the solution of Laplace's three dimensional equation

$$(1.1)\qquad \Delta\phi = 0$$

in the half space $z > 0$, with prescribed values of ϕ at $z = 0$ at the disc $r < a$ and at the annulus $b < r < c$, $a < b$, and with zero values of $\partial\phi/\partial z$ at the remaining parts $a < r < b$, $r > c$ of the $z = 0$ plane.

We pose the problem with symmetry about $z = 0$ and we write the potential $\phi(r,z)$ as the Hankel transform of an arbitrary function $A(\xi)$ in the form

$$(1.2)\qquad \phi(r,z) = \int_0^\infty \xi^{-1}A(\xi)J_0(\xi r)e^{-|z|\xi}d\xi,$$

which is a solution of (1.1) with cylindrical symmetry, and is symmetrical about $z = 0$. $A(\xi)$ is to be determined from the boundary conditions on the plane $z = 0$, namely

$$\phi(r,0) = V_1 \qquad 0 < r < a$$

$$\left.\frac{\partial\phi(r,z)}{\partial z}\right|_{z=0} = 0 \qquad a < r < b$$

$$\phi(r,0) = V_3 \qquad b < r < c$$

$$\left.\frac{\partial\phi(r,z)}{\partial z}\right|_{z=0} = 0 \qquad c < r,$$

where the given values V_1 and V_3 are not depending on r and have opposite signs. The above mixed boundary value problem leads to the quadruple integral equations with unknown $A(\xi)$:

$$(1.3a)\qquad \int_0^\infty \xi^{-1}A(\xi)J_0(\xi r)d\xi = V_1 \qquad 0 < r < a$$

$$(1.3b)\qquad \int_0^\infty A(\xi)J_0(\xi r)d\xi = 0 \qquad a < r < b$$

$$(1.3c)\qquad \int_0^\infty \xi^{-1}A(\xi)J_0(\xi r)d\xi = V_3 \qquad b < r < c$$

$$(1.3d)\qquad \int_0^\infty A(\xi)J_0(\xi r)d\xi = 0 \qquad c < r.$$

As is well known, these four equations for $A(\xi)$ can be reduced to a single Fredholm equation for a function related to $A(\xi)$, from which A, and hence ϕ of (1.2), can be obtained. It is not possible to solve this Fredholm equation explicitly. Therefore we investigate it for limits of some combinations of the parameters a, b and c. That is, we write

$$(1.4)\qquad a = \varepsilon b, \quad c = (1+\mu)b, \quad 0 < \varepsilon < 1, \ \mu > 0$$

and we consider the Fredholm equation for $\varepsilon \to 0$ or $\varepsilon \to 1$ and for $\mu \to 0$ or $\mu \to \infty$ and for combinations of ε and μ limits. It is possible, by scaling of r, to fix the value of b.

We give two different methods for obtaining the Fredholm equation; each method gives a different type of equation for a different function. The first method can be found in the literature, see, for instance, GUPTA & CHATURVEDI (1970) and COOKE (1972). In both papers a more general set of equations is considered, that is, with more general functions in the right-hand sides of (1.3) and with Bessel functions $J_\nu(\xi r)$ with general order ν.

Cooke solved the equations by using the so-called Erdélyi-Kober operators, which are in fact generalizations of the operators associated with the Abel integral equations. In this paper we do not use these operators. We translate the results of GUPTA & CHATURVEDI (1970) for the special case considered here and we modify the resulting inhomogeneous Fredholm equation somewhat. This will be done in Section 2. In Section 3 we derive a homogeneous Fredholm equation, which appears to be a good starting point for obtaining asymptotic solutions. In Section 4 we give a Fredholm equation of the first kind and the asymptotic expansions (based on the results of Section 3) will be considered in Section 5. It turns out that the asymptotic solutions for the several cases can be given in terms of the complete elliptic integral of the first kind. The methods are based on the papers by SPENCE (1970), (1971).

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2. AN INHOMOGENEOUS FREDHOLM EQUATION OF THE SECOND KIND

In this section we summarize the results of GUPTA & CHATURVEDI (1970) for the solution of (1.3).

It is convenient to introduce the functions

$$V(r) = \phi(r,0), \qquad j(r) = -\left.\frac{\partial\phi(r,z)}{\partial z}\right|_{z=0} .$$

We consider the restrictions V_i, j_i of these functions on the four intervals

$$I_1 = (0,a), \quad I_2 = (a,b), \quad I_3 = (b,c), \quad I_4 = (c,\infty).$$

Then V_1, V_3, j_2, j_4 are known and V_2, V_4, j_1, j_3 are to be determined. We have

$$\text{(2.1)} \qquad \begin{aligned} V(r) &= \int_0^\infty \xi^{-1} A(\xi) J_0(\xi r)\, d\xi, \\ j(r) &= \int_0^\infty A(\xi) J_0(\xi r)\, d\xi. \end{aligned}$$

Hankel inversion gives

$$\text{(2.2)} \qquad A(\xi) = \xi \int_0^a r j_1(r) J_0(\xi r)\, dr + \xi \int_b^c r j_3(r) J_0(\xi r)\, dr$$

and substitution of this in the first equation of (2.1) yields

$$\text{(2.3)} \qquad V(\rho) = \int_0^a r j_1(r) L(r,\rho)\, dr + \int_b^c r j_3(r) L(r,\rho)\, dr,$$

where

$$(2.4)\qquad L(r,\rho) = \int_0^\infty J_0(xr)J_0(x\rho)\,dx = \frac{2}{\pi(x+y)}\,K\left(\frac{2\sqrt{xy}}{x+y}\right) = \frac{2}{\pi}\int_0^{\min(r,\rho)} \frac{ds}{\sqrt{(x^2-s^2)(y^2-s^2)}}\,.$$

K is the well-known elliptic integral of the first kind. A first result of GUPTA & CHATURVEDI (1970) is (see their formula (5.7))

$$(2.5)\qquad j_1(r) = \frac{2}{\pi\sqrt{a^2-r^2}}\left[V_1 - \int_b^c \frac{t j_3(t)\sqrt{t^2-a^2}}{t^2-r^2}\,dt\right]$$

which expresses the unknown j_1 in terms of the unknown j_3 and from which follows the type of singularity of $j_1(r)$ for $r \to a$. We have

$$j_1(r) = \frac{2V_1}{\pi\sqrt{a^2-r^2}}\,[1+\mathcal{O}(1)], \qquad r \to a.$$

It is interesting to compare this result with the case of the absent annulus. The known solution of this much simpler problem is

$$(2.6)\qquad \phi_0(r,z) = \frac{2V_1}{\pi}\arcsin\,\zeta(a), \qquad \zeta(t) = \frac{2t}{\sqrt{z^2+(t+r)^2}+\sqrt{z^2+(t-r)^2}}\,,$$

from which follows that

$$V_0(r) = \phi_0(r,0) = \begin{cases} V_1, & 0 \le r \le a \\ \dfrac{2V_1}{\pi}\arcsin(a/r), & a \le r \end{cases}$$

$$j_0(r) = -\left.\frac{\partial\phi_0(r,z)}{\partial z}\right|_{z=0} = \begin{cases} 0, & 0 \le r < a \\ \dfrac{2}{\pi}\dfrac{V_1}{\sqrt{a^2-r^2}}\,, & a < r. \end{cases}$$

Hence, in our case, the charge density $j_1(r)$ has the same type of singularity at $r = a$ as $j_0(r)$ of the simpler problem. It follows from (2.5) that the difference is just an integral over the annulus. More analogues between the two cases will follow from the results of Section 3.

The next result in GUPTA & CHATURVEDI (1970) (see their equation (5.17)) is the inhomogeneous Fredholm equation

$$(2.7)\qquad m(s) = f(s) - \left(\frac{2}{\pi}\right)^2 \int_b^c m(u)K(u,s)\,du, \qquad b < s < c,$$

where

$$(2.8)\qquad m(s) = \int_s^c \frac{rj_3(r)}{\sqrt{r^2-s^2}}\,dr, \qquad b < s < c,$$

$$f(s) = \frac{s}{\sqrt{s^2-b^2}}\left[V_3 - \frac{2}{\pi} V_1 \arcsin \frac{a}{b}\right] + \frac{2}{\pi} V_1 \arcsin \left[\frac{a}{b}\sqrt{\frac{s^2-b^2}{s^2-a^2}}\right],$$

$$K(u,s) = \frac{us}{2(u^2-s^2)}\left[\sqrt{\frac{u^2-b^2}{s^2-b^2}}\,L(u) - \sqrt{\frac{s^2-b^2}{u^2-b^2}}\,L(s)\right],$$

$$L(t) = \frac{1}{t}\,\ell n\,\frac{(t+b)(t-a)}{(t-b)(t+a)}\,.$$

The kernel $K(u,s)$ of (2.7) is symmetric in u and s and it is singular at the boundary of the square $b \le s \le c$, $b \le u \le c$ in the (s,u)-plane. By inverting the Abel equation (2.8), see (2.11), we obtain

$$(2.9)\qquad rj_3(r) = -\frac{2}{\pi}\frac{d}{dr}\int_r^c \frac{sm(s)}{\sqrt{s^2-r^2}}\,ds, \qquad b < r < c,$$

from which the functions in (2.5), (2.3) and (2.2) can be computed.

It is possible to give a Fredholm equation of the first kind for $j_3(r)$ itself. This will be done in Section 4. To conclude this section we modify (2.7) into a somewhat simpler form.

To do so, we remark that a necessary condition for convergence of the integral in (2.7), at $u = b$ is

$$\lim_{u \downarrow b} m(u)\sqrt{u^2-b^2} = 0.$$

Hence, by multiplying (2.7) by $\sqrt{s^2-b^2}$ and taking the limit $s \downarrow b$ we obtain

$$\frac{4}{\pi^2}\int_b^c \frac{um(u)}{\sqrt{u^2-b^2}} L(u)du = s[V_3 - \frac{2V_1}{\pi} \arcsin \frac{a}{b}].$$

Using this in (2.7) we find

$$\frac{m(s)}{\sqrt{s^2-b^2}} = \frac{2V_1}{\pi\sqrt{s^2-b^2}} \text{arc}\sin[\frac{a}{b}\sqrt{\frac{s^2-b^2}{s^2-a^2}}] +$$

(2.10)

$$\frac{4}{\pi^2}\int_b^c \frac{m(u)}{\sqrt{u^2-b^2}} \frac{su}{2(s^2-u^2)} [L(u) - (L(s)]du.$$

A similar kernel occurs in the next section (see (3.21) or (3.23)).

The Abel equations used in this paper are of the form

$$g(x) = \int_a^x \frac{f(t)}{\sqrt{x^2-t^2}} dt$$

(2.11a) $\qquad a < x < b$

$$g(x) = \int_x^b \frac{f(t)}{\sqrt{t^2-x^2}} dt$$

with the respective solutions

$$f(t) = \frac{2}{\pi}\frac{d}{dt}\int_a^t \frac{xg(x)}{\sqrt{t^2-x^2}} dx$$

(2.11b) $\qquad a < t < b$

$$f(t) = -\frac{2}{\pi}\frac{d}{dt}\int_t^b \frac{xg(x)}{\sqrt{t^2-x^2}} dx$$

where g and g' are supposed to be continuous on (a,b).

3.1. Homogeneous Fredholm equations of the second kind

We write the unknown function A of (1.2) and (1.3) in the form

$$A(\xi) = \int_0^\infty B(t)\sin\xi t\,dt, \tag{3.1}$$

where B is to be determined. The restriction of B on the four intervals (0,a), etc., are labelled $B_1(t)$, etc. In this section we show that B_1 and B_4 vanish identically on (0,a), (c,∞) respectively, and that B_2 and B_3 can be obtained by solving a homogeneous Fredholm equation of the second kind.

By using (3.1) and (see GRADSHTEYN & RYZHIK (1965), 6.752(1))

$$\int_0^\infty \xi^{-1}J_0(\xi r)e^{-z\xi}\sin\xi t\,d\xi = \arcsin\zeta(t),$$

where $\zeta(t)$ is defined in (2.6), we can transform (1.2) into

$$\phi(r,z) = \int_0^\infty B(t)\arcsin\zeta(t)\,dt. \tag{3.2}$$

This integral reduces to a finite integral over (a,c) since, as will be shown, B vanishes outside this interval. Moreover, (3.2) does not contain the Bessel function $J_0(x)$. This simplifies the computations when ϕ is to be evaluated numerically. We proceed with the construction of the equations for the functions B_i.

From (3.2), or by substituting (3.1) into (2.1), we obtain

$$\begin{aligned} V(r) &= \int_0^\infty B(t)H(r,t)\,dt \\ j(r) &= \int_0^\infty B(t)G(r,t)\,dt \end{aligned} \tag{3.3}$$

with

$$H(r,t) = \begin{cases} \frac{1}{2}\pi & \text{if } t \ge r > 0 \\ \arcsin t/r & \text{if } 0 \le t \le r \end{cases}, \tag{3.4}$$

$$G(r,t) = \begin{cases} 0 & \text{if } r > t \ge 0 \\ 1/\sqrt{t^2-r^2} & \text{if } t > r > 0 \end{cases} \tag{3.5}$$

It follows that

$$(3.6)\qquad H(r,t) = \int_0^{\min(t,r)} \frac{dy}{\sqrt{r^2-y^2}}$$

Substituting this formula into the first of (3.3) and interchanging the order of integration, we obtain

$$(3.7)\qquad V(r) = \int_0^r \frac{F(y)\,dy}{\sqrt{r^2-y^2}}\ ,\qquad r > 0,$$

with

$$(3.8)\qquad F(y) = \int_y^\infty B(t)\,dt,\qquad y > 0.$$

This result also follows from partial integration of (3.2) at $z = 0$.

Inverting the Abel equation (3.7) we find, using (2.11),

$$(3.9)\qquad F(y) = \frac{2}{\pi}\frac{d}{dy}\int_0^y \frac{rV(r)}{\sqrt{y^2-r^2}}\,dr,\qquad y > 0.$$

When we take $y < a$ in this integral then $V(r) = V_1$, see (1.3a), and thus we obtain

$$(3.10)\qquad F(y) = \frac{2V_1}{\pi}\frac{d}{dy}\int_0^y \frac{r\,dr}{\sqrt{y^2-r^2}} = \frac{2V_1}{\pi} = \text{constant},\qquad 0 < y < a.$$

Hence

$$B_1(t) = -\frac{dF(t)}{dt} = 0,\qquad 0 < t < a.$$

Furthermore, (1.3d), (3.3) and (3.5) give for $r > c$

$$\int_0^\infty B(t)G(r,t)\,dt = \int_r^\infty B_4(t)\frac{dt}{\sqrt{t^2-r^2}} = 0,$$

hence

$$B_4(t) = 0 \quad \text{on } (c,\infty).$$

Consequently, using (3.8) we find

$$F(y) = 0, \quad y > c.$$

Incidentally it follows that (3.1) can be replaced by

$$A(\xi) = \int_a^c B(t)\sin\xi t\,dt \tag{3.11}$$

and in (3.2) the integral can be taken over (a,c) as well.

Up to now we used the boundary conditions (1.3a) and (1.3d), with the results $B_1 = B_4 = 0$. As an extra result we obtain from (3.8) and (3.10)

$$\int_a^c B(t)\,dt = \frac{2}{\pi} V_1, \tag{3.12}$$

which gives a normalization for the functions B_2 and B_3.

For $a < r < c$ we have

$$V(r) = \int_a^r B(t)\arcsin(t/r)\,dt + \frac{1}{2}\pi\int_r^c B(t)\,dt. \tag{3.13}$$

Hence, in view of (3.12), we have

$$V_2(r) = V_1 - \int_a^r B_2(t)\arccos(t/r)\,dt, \quad a < r < b,$$

$$V_3(r) = \int_a^r B(t)\arcsin(t/r)\,dt + \frac{1}{2}\pi\int_r^c B_3(t)\,dt, \quad b < r < c,$$

where we used (3.12). Since $V_2(b) = V_3$, a second normalization follows from

$$\int_a^b B_3(t)\arccos(t/b)\,dt = V_1 - V_3. \tag{3.14}$$

Approximations for $B_2(t)$, given in Section 5, have a fixed sign on (a,b); in that event they have the sign of $V_1 - V_3$, as follows from (3.14). Thus $V_2(r)$ is a monotonous function on (a,b). For $r > c$ we have

$$V_4(r) = V_3 + \int_a^c B(t)[\arcsin(t/r) - \arcsin(t/c)]\,dt, \tag{3.15}$$

which equals V_3 at $r = c$.

Differentiating (3.13) with respect to r, $b < r < c$, we find

$$\int_a^r \frac{tB(t)}{\sqrt{r^2-t^2}}\,dt = 0, \qquad b < r < c. \tag{3.16}$$

Finally, using (3.3) and (1.3b), we obtain

$$\int_r^c \frac{B(t)}{\sqrt{t^2-r^2}}\,dt = 0, \qquad a < r < b. \tag{3.17}$$

From (3.16) and (3.17) and the normalization in (3.14) the functions B_2 and B_3 can be obtained. Let us first write (3.16) and (3.17) in the form

$$\begin{aligned} \int_b^r \frac{tB_3(t)}{\sqrt{r^2-t^2}}\,dt &= -\int_a^t \frac{tB_2(t)}{\sqrt{r^2-t^2}}\,dt, \qquad b < r < c, \\ \int_r^b \frac{B_2(t)}{\sqrt{t^2-r^2}}\,dt &= -\int_b^c \frac{B_3(t)}{\sqrt{t^2-r^2}}\,dt, \qquad a < r < b, \end{aligned} \tag{3.18}$$

giving two Abel equations. Inversion yields

$$\begin{aligned} B_2(t) &= \frac{-2t}{\pi\sqrt{b^2-t^2}} \int_b^c \frac{B_3(\tau)\sqrt{\tau^2-b^2}}{\tau^2-t^2}\,d\tau \qquad a < t < b, \\ B_3(t) &= \frac{-2}{\pi\sqrt{t^2-b^2}} \int_a^b \frac{\tau B_2(\tau)\sqrt{b^2-\tau^2}}{t^2-\tau^2}\,d\tau \qquad b < t < c. \end{aligned} \tag{3.19}$$

Substituting these relations into each other we obtain two homogeneous Fredholm equations of the second kind

$$y(t) = \frac{4}{\pi^2}\int_a^b y(s)K_2(s,t;b,c)\,ds, \qquad a < t < b, \tag{3.20}$$

with

$$y(t) = B_2(t)\sqrt{b^2-t^2}$$

$$(3.21)\qquad K_2(s,t;b,c) = \frac{1}{2(s^2-t^2)}\left[t\,\ell n\frac{c-s}{c+s}\,\frac{b+s}{b-s} - s\,\ell n\frac{c-t}{c+t}\,\frac{b+t}{b-t}\right]$$

and

$$(3.22)\qquad z(t) = \frac{4}{\pi^2}\int_b^c z(s)K_3(s,t;a,b)\,ds, \qquad b < t < c,$$

with

$$(3.23)\qquad \begin{aligned} z(t) &= B_3(t)\sqrt{t^2-b^2} \\ K_3(s,t;a,b) &= \frac{1}{2(t^2-s^2)}\left[s\,\ell n\frac{s+b}{s-b}\,\frac{s-a}{s+a} - t\,\ell n\frac{t+b}{t-b}\,\frac{t-a}{t+a}\right]. \end{aligned}$$

The normalization for the solutions y and z of these equations can be obtained from (3.15) and, for instance, from

$$(3.24)\qquad z(t) = -\frac{2}{\pi}\int_a^b \frac{\tau y(\tau)}{t^2-\tau^2}\,d\tau.$$

3.2. Similarity aspects of the equations and their solutions

In Section 5, asymptotic solutions of the equations (3.20) and (3.22) are given for $\varepsilon \to 0$, $\varepsilon \to 1$, $\mu \to 0$, $\mu \to \infty$, where ε and μ are defined in (1.4). In the asymptotic analysis we can fix the parameter b. This follows from the following lemma.

LEMMA 1. *Let* $y(t;a,b,c)$ *be a solution of* (3.20). *Then* $\bar{y}(t) = \nu y(\lambda t;\lambda a,\lambda b,\lambda c)$ *is also a solution, where* ν *and* λ *are arbitrary real numbers* ($\lambda \neq 0$).

PROOF. Let $\bar{y}(\sigma) = \bar{y}(\sigma;\alpha,\beta,\gamma)$ denote the solution of the equation

$$\bar{y}(\tau) = \int_\alpha^\beta \bar{y}(\sigma)K_2(\sigma,\tau;\beta,\gamma)\,d\sigma,$$

where $\alpha = \lambda a$, $\beta = \lambda b$, $\gamma = \lambda c$ ($\lambda \in \mathbb{R}$, $\lambda \neq 0$). Writing $\sigma = \lambda s$, $\tau = \lambda t$, we deduce that $\bar{y}(\lambda t)$ is a solution of (3.20), which is denoted by $y(t;a,b,c)$. This proves the lemma. □

There is also similarity between the equations (3.20) and (3.22), and between their solutions. The next lemma shows that, up to a transformation, (3.20) and (3.22) are the same.

LEMMA 2. *Let* $z(t;a,b,c)$ *denote a solution of* (3.22). *Then*

$$y(t;a,b,c) = \frac{\nu}{t}\, z\left(\frac{b\beta}{t};\frac{b\beta}{c},\beta,\frac{b\beta}{a}\right), \tag{3.25}$$

where ν *and* β *are arbitrary real numbers* $(\beta \neq 0)$, *is a solution of* (3.20).

PROOF. Let us write in (3.20) $s = \lambda b/\sigma$, $t = \lambda b/\tau$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Then we obtain

$$u(\tau) = \frac{4}{\pi^2}\int_{\lambda}^{\lambda b/a} u(\sigma)\,\frac{1}{2(\tau^2-\sigma^2)}\left[\sigma\,\ell n\,\frac{\sigma-\alpha}{\sigma+\alpha}\,\frac{\sigma+\lambda}{\sigma-\lambda} - \tau\,\ell n\,\frac{\tau-\alpha}{\tau+\alpha}\,\frac{\tau+\lambda}{\tau-\lambda}\right]d\tau,$$

where $\alpha = \lambda b/c$ and $u(\tau)$ is given by $\tau u(\tau) = y(\lambda b/\tau;a,b,c)$. With $\lambda = \beta$, $\gamma = \lambda b/a$ this equation is of the form of (3.22), i.e., $u(\tau) = z(\tau;\alpha,\beta,\gamma)$, from which the lemma follows. □

Lemma 2 is useful in the investigation of the asymptotic expansions. For instance, if we consider (3.20) with $a = \varepsilon b$, $\varepsilon \to 0$, then we cover also the analysis for (3.22) with $c/b \to \infty$.

A remarkable situation occurs when the parameters a, b and c are such that

$$b^2 = ac. \tag{3.26}$$

In this situation we can choose the free parameter β such that (3.25) becomes (with $\beta = b$)

$$y(t;a,b,c) = \frac{\nu}{t}\, z(b^2/t;a,b,c).$$

Proceeding with this special situation we deduce that B_2 and B_3 are related to each other according

$$\tau^2 B_3(\tau) = \mu B_2(b^2/\tau)$$

for some μ not depending on τ. Substituting this in the second equation of (3.19), we find the rather simple equation

$$(3.27) \qquad g(\sigma) = \frac{2}{\pi} \lambda\sigma^2 \int_{a/b}^{1} \frac{g(\tau)}{1-\sigma^2\tau^2} d\tau,$$

with

$$g(\sigma) = \sigma B_2(b\sigma)\sqrt{1-\sigma^2}$$

under the assumption (3.26); λ is some real parameter. An explicit solution of (3.27) is not known to us. However, the equation

$$(3.28) \qquad g_0(\sigma) = \frac{2}{\pi} \sigma^2 \int_{0}^{1} \frac{g_0(\tau)}{1-\tau^2\sigma^2} d\tau$$

is considered in Section 5. The solution of this equation is

$$(3.29) \qquad g_0(\sigma) = \frac{\sigma}{\sqrt{1-\sigma^2}} \ln \frac{1+\sigma}{1-\sigma} .$$

It is expected that the solution of (3.27) can be described in terms of $g_0(\sigma)$ when small values of $a/b = b/c$ are considered in a perturbation analysis. Also, it is expected that λ of (3.27) will approach 1 in that event.

4. A FREDHOLM EQUATION OF THE FIRST KIND

In this section we derive an equation for the function $j_3(r)$ introduced in Section 3. In that Section the problem was solved by the equation (2.7) (or (2.10)) for the function $m(s)$ defined in terms of $j_3(r)$ in (2.8). It is useful to have an equation for $j_3(r)$ itself.

Starting point is (2.5), which is written as

$$(4.1) \qquad j_1(\rho) = [\tfrac{2}{\pi} V_1 + h(\rho)]/\sqrt{a^2-\rho^2}, \qquad 0 < \rho < a,$$

defining the function $h(\rho)$ as

$$(4.2) \qquad h(\rho) = -\frac{2}{\pi}\int_b^c \frac{rj_3(r)\sqrt{r^2-a^2}}{r^2-\rho^2}\,dr.$$

From (2.3) it follows that

$$(4.3) \qquad V(r) = \frac{2}{\pi} V_1 \int_0^a \frac{\rho}{\sqrt{a^2-\rho^2}} L(r,\rho)d\rho + \int_0^a \frac{\rho h(\rho)}{\sqrt{a^2-\rho^2}} L(r,\rho)d\rho + \int_b^c \rho j_3(\rho)L(r,\rho)d\rho$$

with $L(r,\rho)$ given in (2.4). The first integral is easily computed. It reduces to

$$\frac{2}{\pi} V_1 \int_0^a \frac{\rho}{\sqrt{a^2-\rho^2}} \int_0^\infty J_0(\xi\rho)J_0(\xi r)d\xi d\rho =$$

$$\frac{2}{\pi} V_1 \int_0^\infty J_0(\xi r) \int_0^a \frac{\rho}{\sqrt{a^2-\rho^2}} J_0(\xi\rho)d\rho d\xi =$$

$$\frac{2}{\pi} V_1 \int_0 J_0(\xi r)\frac{\sin\xi a}{\xi}\,d\xi = \begin{cases} V_1 & 0 \le r \le a \\ \frac{2}{\pi} V_1 \arcsin(a/r) & \text{if } r \ge a \end{cases}$$

where we used well-known relations for the Bessel functions (see, for instance, GRADSHTEYN & RYZHIK (1965, 6.554(2) and 6.693(1)).

The second integral of (4.3) gives

$$-\frac{2}{\pi}\int_b^c tj_3(t)\sqrt{t^2-a^2}\int_0^a \frac{\rho L(r,\rho)}{\sqrt{a^2-\rho^2}(t^2-\rho^2)}\,d\rho dt =$$

$$-\frac{2}{\pi}\int_b^c tj_3(t)N(t,r)dt,$$

where for $r > a$

$$N(t,r) = \frac{2}{\pi}\sqrt{t^2-a^2}\int_0^a \frac{\rho}{\sqrt{a^2-\rho^2}\,(t^2-\rho^2)}\int_0^\rho \frac{ds}{\sqrt{(r^2-s^2)(\rho^2-s^2)}}\,d\rho =$$

$$\frac{2}{\pi}\sqrt{t^2-a^2}\int_0^a \frac{1}{\sqrt{r^2-s^2}}\int_s^a \frac{\rho\,d\rho}{\sqrt{(a^2-\rho^2)(\rho^2-s^2)}\,(t^2-\rho^2)}\,ds = \tag{4.4}$$

$$\int_0^a \frac{ds}{\sqrt{(r^2-s^2)(t^2-a^2)}}\,.$$

So, for $b < r < c$, (4.3) can be written as

$$\int_b^c \rho j_3(\rho)[L(r,\rho)-\tfrac{2}{\pi}N(r,\rho)]d\rho = V_3 - \frac{2}{\pi}V_1 \text{ arcsin } a/r.$$

Using (4.4) and (2.4), we obtain for $r > a$, $\rho > a$

$$M(r,\rho) = L(r,\rho) - \frac{2}{\pi}N(r,\rho) = \frac{2}{\pi}\int_a^{\min(r,\rho)} \frac{ds}{\sqrt{(r^2-s^2)(\rho^2-s^2)}}\,,$$

which is an incomplete elliptic integral. The integral equation for j_3 thus becomes

$$\frac{2}{\pi}\int_b^c \rho j_3(\rho)M(r,\rho)d\rho = V_3 - \frac{2}{\pi}V_1 \text{ arcsin}(a/r), \quad b < r < c, \tag{4.5}$$

in which the symmetric kernel $M(r,\rho)$ is singular at $r = \rho$. The behaviour of $M(r,\rho)$ for $r \to \rho$ is given by

$$M(r,\rho) = -\frac{1}{2}\ln|r^2-\rho^2| + O(1).$$

In terms of standard elliptic functions we have for $r < \rho$

$$M(r,\rho) = \frac{1}{\rho}F(\phi,r/\rho), \quad \sin\phi = \frac{\rho}{r}\sqrt{(r^2-a^2)/(\rho^2-a^2)},$$

where $F(\phi,k)$ is the well-known elliptic integral of the first kind

$$F(\phi,k) = \int_0^\phi \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \int_0^{\sin\phi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}\,.$$

For $a = 0$, M reduces to the complete elliptic integral.

5. ASYMPTOTIC EXPANSIONS OF SOLUTIONS

In this section we consider limit solutions of the equations (3.20) for $a \to 0$, $a \to b$, $c \to b$, $c \to \infty$. From Lemma 1 of Section 3.1 it follows that we can fix the parameter b. From Lemma 2 we can derive equivalent results for the solution of (3.22). However, in some cases it is better to use the second of (3.19). After normalizing the asymptotic solution of (3.20) by using (3.14) the proper normalization of B_3 is thus obtained at the same time.

Let us introduce new parameters ε, μ by writing

$$(5.1)\qquad a = \varepsilon b, \quad c = (1+\mu)b, \quad 0 < \varepsilon < 1,\ \mu > 0.$$

Then (3.20) becomes with $u(\xi) = y(b\xi)$

$$(5.2)\qquad u(\xi) = \frac{2}{\pi^2}\int_\varepsilon^1 \frac{u(\eta)}{\eta^2-\xi^2}\left[\xi \,\ell n \frac{1+\mu-\eta}{1+\mu+\eta}\,\frac{1+\eta}{1-\eta} - \eta\,\ell n \frac{1+\mu-\xi}{1+\mu+\xi}\,\frac{1+\xi}{1-\xi}\right]d\eta .$$

We can consider this equation in the limits $\varepsilon \to 0$, $\varepsilon \to 1$, $\mu \to 0$, $\mu \to \infty$, where ε and μ limits may be combined with one another. Limiting forms of (5.2) appear to be the two equations

$$(5.3)\qquad \begin{aligned} v(\xi) &= \frac{2}{\pi^2}\int_0^1 \frac{v(\eta)}{\eta^2-\xi^2}\left[\xi\,\ell n\frac{1+\eta}{1-\eta} - \eta\,\ell n \frac{1+\xi}{1-\xi}\right]d\eta, \\ w(\xi) &= \frac{1}{\pi^2}\int_0^1 \frac{w(\eta)}{\xi-\eta}\left[\ell n\ \xi - \ell n\ \eta\right]d\eta, \end{aligned}$$

with the respective solutions (see SPENCE (1971))

$$(5.4)\qquad v(\xi) = \frac{\xi}{\sqrt{1-\xi^2}}\,\ell n\,\frac{1+\xi}{1-\xi}, \qquad w(\xi) = \xi^{-\frac{1}{2}}\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;1-\xi).$$

The Gaussian hypergeometric function is in fact the elliptic integral K, according to $K(m) = \frac{1}{2}\pi\ {}_2F_1(\frac{1}{2},\frac{1}{2};1;m)$.

<u>REMARK</u>. SPENCE (1971) introduced for the proof of the v-function the operator

$$Tf(x) = \frac{2}{\pi}\int_0^1 \frac{f(y)dy}{1-y^2x^2}$$

and he remarked that $Tv(\xi) = v(\xi)/\xi^2$ and $K = T^2$, where K is defined by writing the first equation of (5.3) as $Kv = v$. His proof for the w-function is based on Wiener-Hopf techniques and Mellin transforms. A simpler proof is possible, however, by introducing as for the v-case a similar operator for the w-case. Let us define the operator

$$Sf(x) = \frac{1}{\pi}\int_0^1 \frac{f(y)}{1-xy}\,dy.$$

A well-known result for the hypergeometric functions is $Sf = f$ with $f(x) = (1-x)^{-\frac{1}{2}}\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)$. Furthermore, it is easily shown that S^2 is related to the integral operator of the second equation of (5.3).

By giving an example we will show how the analysis proceeds. We consider (5.2) for $\mu \to 0$, $\varepsilon \to 1$. In the analysis we restrict ourselves to first order approximations. It is possible to give approximations of higher order. For more information on this point see SPENCE (1971).

EXAMPLE. $\mu \to 0$, $\varepsilon \to 1$, $(1-\varepsilon)/\mu$ bounded.
The transformations

$$\eta = 1 - \mu q, \quad \xi = 1 - \mu p, \quad u(\xi) = (1-\mu p) = f(p)$$

transform (5.2) into

$$f(p) = \frac{1}{\pi^2}\int_0^\gamma \frac{f(q)}{(p-q)[1-\frac{1}{2}\mu(p+q)]}\Big[(1-\mu p)\,\ell n\frac{1+q}{q}\,\frac{2-\mu q}{2+\mu+\mu q} - (1-\mu q)\,\ell n\frac{1+p}{p}\,\frac{2-\mu p}{2+\mu+\mu p}\Big]dq,$$

with $\gamma = (1-\varepsilon)/\mu$. If γ is bounded then p, q are bounded and we obtain the reduced equation

$$f_0(p) = \frac{1}{\pi^2}\int_0^\gamma \frac{f_0(q)}{p-q}\,\ell n\,\frac{1+q}{q}\,\frac{p}{1+p}\,dq.$$

A further transformation

$$\frac{q}{1+q} = \delta t, \qquad \frac{p}{1+p} = \delta s, \qquad \delta = \frac{\gamma}{1+\gamma}, \qquad w(t) = \frac{f_0\left(\frac{\delta t}{1-\delta t}\right)}{1-\delta t}$$

gives

$$w(s) = \frac{1}{\pi^2}\int_0^1 w(t)\,\frac{\ell n\, t - \ell n\, s}{t-s}\,dt$$

of which the solution is given in (5.3). In terms of the original variables we have

$$y(t) = \frac{\lambda}{\sqrt{(c-t)(b-t)}}\; {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\frac{c-b}{b-a}\,\frac{t-a}{c-t}\right),$$

where λ is to be determined by normalization. To compute λ we use (3.14) and $y(t) = B_2(t)\sqrt{b^2-t^2}$. This results in the equation

$$\lambda\int_a^b \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{c-b}{b-a}\,\frac{t-a}{c-t}\right)\arccos(t/b)}{(b-t)\sqrt{(b+t)(c-t)}}\,dt = V_1 - V_3,$$

again under the conditions $\varepsilon \to 1$, $\mu \to 0$, such that $\gamma = (1-\varepsilon)/\mu$ is bounded. Using

$$\arccos(t/b) = 2\arcsin\sqrt{\frac{b-t}{2b}}$$

and writing $s = \frac{1}{\gamma}\,\frac{t-a}{c-t}$ we obtain the equation

$$\frac{\lambda\sqrt{c-a}}{c-b}\int_0^1 \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;s\right)\arcsin\sqrt{\frac{1-\varepsilon}{2}\,\frac{1-s}{1+\gamma s}}}{(1-s)\sqrt{1+\varepsilon+\gamma s(2+\mu)}}\,ds = V_1 - V_3.$$

For $\varepsilon \to 1$, $\mu \to 0$ we obtain

$$\text{(5.5)} \qquad \lambda = \frac{2(V_1-V_3)\sqrt{b(c-a)/(b-a)}}{\pi\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;(b-a)/(c-a)\right)}$$

where we used the known result

$$\frac{1}{\pi}\int_0^1 (1-x)^{-\frac{1}{2}}(1-zx)^{-1}\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;x)\,dx = (1-z)^{-\frac{1}{2}}\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;z)$$

$$= (1-z)^{-1}\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;z/(z-1)),$$

which is also mentioned in the above remark.

Hence, a first approximation for the function B_2 on (a,b) is given by

$$(5.6)\qquad B_2(t) = \frac{2(V_1-V_3)}{\pi(b-t)}\left[\frac{b(c-a)}{(b-a)(c-t)(b+t)}\right]^{\frac{1}{2}} \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\frac{c-b}{b-a}\,\frac{t-a}{c-t})}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\frac{b-a}{c-a})}.$$

Note that B_2 has a non-integrable singularity at $t = b$. As a consequence, some integrals of Section 3 have to be interpreted as Cauchy principal value integrals. The function B_3 has a similar singular behaviour at the left end point of the interval (b,c). We compute B_3 by using the second of (3.19), again under the conditions $\varepsilon \to 1$, $\mu \to 0$, γ bounded. We obtain as a first approximation

$$(5.7)\qquad B_3(t) = \frac{-\lambda}{t-b}\left[\frac{b-a}{(c-b)(t-a)(t+b)}\right]^{\frac{1}{2}}\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\tfrac{b-a}{c-b}\,\tfrac{c-t}{t-a})$$

where λ is given by (5.5)

Remark that apart from the singularity $1/(t-b)$ in B_2 and B_3, another (logarithmic) singularity at $t = b$ is caused by the ${}_2F_1$ function. It is easy to show that

$$\lim_{t\to 0}\frac{B_2(b-t)}{B_3(b+t)} = -1.$$

This limit is expected for Cauchy principal value integrals as (3.1), (3.2), (3.3), (3.8), (3.12), etc. The functions $y(t)$ and $z(t)$ of (3.20) and (3.22) are integrable at $t = b$.

Other examples can be elaborated in the same way. For instance, the case $\varepsilon \to 1$, $\mu \to \infty$ has the first order approximations

$$B_2(t) = \frac{\lambda}{(b-t)\sqrt{b+t}}\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\tfrac{t-a}{b-a}),$$

$$B_3(t) = \frac{-2\lambda\sqrt{b-a}}{(a-t)(t-b)\sqrt{(t+b)(t-a)}}\ {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\tfrac{b-a}{t-a}),$$

$$\lambda = \frac{(V_1-V_3)b}{\sqrt{b-a}\ \pi}\ .$$

In the case $\varepsilon \to 0$, $\mu \to \infty$, (5.2) reduces to the first equation of (5.3), with solution $v(\xi)$ of (5.4).

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